Inflation-Linked Derivatives

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Abstract

Since the introduction of government-issued bonds linked to inflation indices in many major currencies, a liquid market in inflation-linked swaps and other derivatives has grown. The main interest in buying protection from increases in inflation comes from those with inflation-linked liabilities such as pension funds. On the other hand, there are many groups with income linked to inflation (e.g. retail companies), who are well-placed to sell this protection. Inflation-linked derivatives are a convenient way to acquire the desired inflation exposure. In addition there are now several hybrid products available which can guarantee a real (inflation-floored) return, but which still tap into gains in some other asset (e.g. an equity index).

In this talk we will cover the pricing of inflation derivatives within a correlated Hull-White model. Here we consider the short interest rate and the inflation rate as diffusive processes with mean reversion. For constant volatiltiy there are exact results in closed form for simple options. We price more complex derivatives using Monte Carlo sampling. This method also allows us to introduce generalizations to the model (e.g. local volatility) in order to capture the dependence of market prices on, say, the strike of an inflation caplet.

This model naturally includes hybrid products that involve both interest rates and inflation (e.g. a floating rate-inflation rate swap). We also generalize to a third stochastic process to allow other hybrids. For example, considering an equity index as a stochastic variable, we can price a bond that pays out the maximum of gains in the equity index or the change in a price index.

Contents

1 Background on Inflation

Inflation is a process we are all familiar with: the increase over time of the prices of goods and services, and so in effect the “real value” of money.[1] If prices are not fixed, the value of money will float, usually upwards. The reasons behind inflation are complex and various, and will not be dwelled on here. Rather we will be looking at the financial products now available which give exposure to inflation, and the modelling needed to price inflation-linked derivatives.

In the following we briefly introduce how inflation is measured by price indices, and how real returns can be guaranteed by buying inflation-linked bonds. Then we will describe the young but growing market in inflation-linked derivatives. We then define some models that can be used to price these derivatives: the Jarrow-Yildirim model which models real and nominal rates of return as correlated one-factor stochastic processes, and inflation with a log-normal exchange-rate model; a market model which uses log-normal models for the market-projected level of prices as well as forward interest rates, and finally a two-process short-rate model for interest rates and the inflation rate. This last two-process correlated Hull-White model for inflation is the one we will describe in detail here. We show how to price some simple inflation options, and how to calibrate to market prices. A shortcoming of this model is that it does not capture very well the smile of market prices as a function of strike, and so we generalize the model to one with local volatility. With this generalized model, even the simplest of options must be priced numerically, and we use Monte Carlo sampling to calibrate to market prices of inflation caps and floors. Finally we consider hybrid products. We show how to price an interest-rate inflation hybrid with the two-process Hull-White model, and then how to price an equity-inflation hybrid by adding a third stochastic process.

1.1 Consumer Price Indices

There is a problem in defining inflation: The relative value of any two products/services changes with time. Hence the “value of money”, and therefore the level of inflation, is always subjective to some extent. This is the reason why one often refers to a sample, or basket, of different goods and services to establish the level of inflation. Price indices are defined with respect to different baskets, and calculated by various national agencies in different countries. These indices can then be used to inform government economic policy, monetary policy, wage bargaining, the level of pension increases etc.

The baskets that define a price index tend to be aimed at being representative of the expenditure of a particular type of consumer. For example, in the UK the Office for National Statistics[2] (ONS) measures the retail price index (RPI) which relates the price of a basket of goods and services that represents the total expenditure of a typical household. Note that, as spending patterns change, so will the make-up of the basket, and its relative weights. In addition, the ONS also calculates the consumer price index (CPI) whose main difference to the RPI is that it excludes housing-related expenses such as mortgage payments, council tax. It also has a different measure of typical households, which has an affect on the chosen basket. In the USA, the Bureau of Labor Statistics[3] (BLS) publishes the consumer price index (CPI-U) which is chosen to represent the expenditure of urban residents (87% of population). In France the Institut National de la Statistique et des Études Économiques[4] (INSEE) publishes the consumer price index for all France (FCPI), and importantly for French inflation-linked bonds, also the same index with the exclusion of tobacco (FCPI-x).

[1] Of course, deflation also occurs, rarely for the general level of prices, but quite commonly for individual products, e.g., personal computers.

[2] www.statistics.gov.uk

[3] www.bls.gov

[4] www.insee.fr

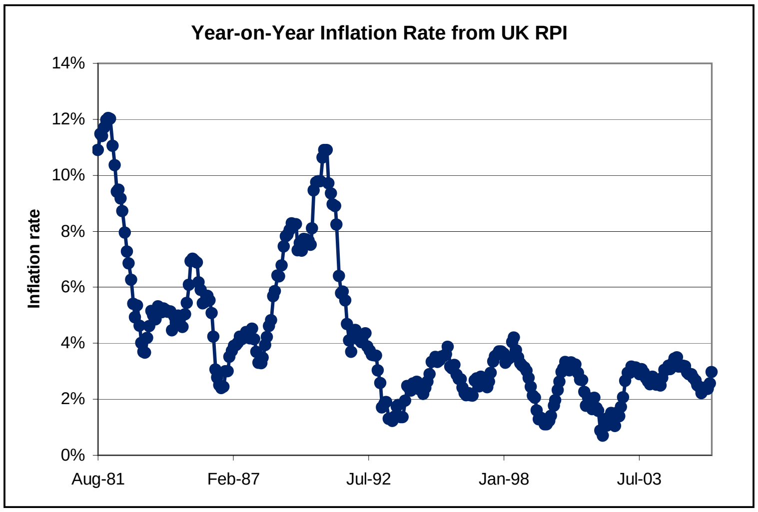

Figure 1: The year-on-year inflation rate $i(t) = -1 + I(t)/I(t-1)$ defined from the UK Retail Price Index $I(t)$.

The consumer price index for the Eurozone (MUICP) is published by Eurostat[5]. In fact the methodology for the UK CPI, and the MUICP are the same, and known as harmonized indices of consumer prices (HICP), which are published by Eurostat for all of the member states of the EU.

Calculating a price index is something of a dark art: Along with the choice of the basket, other choices are the weightings which must be made for different geographical regions, or different brands of the same product, geometric or arithmetic means, and how to allow for improvements in quality of products (e.g. a new car today typically has more features than a new car ten years ago). We can take as illustration the event in September 2000 when the BLS revised the CPI-U due to an error in the calculation of quality adjustments for reported changes of air-conditioning equipment in rental properties. Such an apparently small detail led to a revision in the CPI-U for August 2000 from 172.7 to 172.8.

We end this section with the reminder that price indices are not the only way we might want to measure the changing value of money. For instance, people may value money by what they aspire to buy (sports cars), rather than what they actually do spend money on (gas bills). Or by the (easily measurable) average earnings (cf. UK debate on state pension linkage). Or perhaps by our neighbour's earnings...

1.2 "Real" returns from inflation-linked bonds

Imagine we wanted to celebrate an important anniversary in one year's time, and we will want 100 bottles of champagne. We could buy the bottles now, but this would cost us money which would otherwise earn us interest for six months. On the other hand we might be reasonably worried that the price of champagne in one year will have gone up by more than the interest on our savings. In order to protect against champagne inflation, without losing interest on our money, we could exchange x bottles of champagne with someone now for the promise of 100 bottles a year later. This gives us a real champagne rate of return of $(100-x)/x$. Note, such real returns could in principle exist in an economy with no money (bartering only).

Of course, in our economy there is not a liquid market in real returns of every product and service. In addition, we need to use money to value things. However, we can guarantee a

[5]ec.europa.eu/eurostat/

real return, fixed to a price index, by buying a bond which has coupon and capital payments linked to the said price index. So, for example, if we buy a zero-coupon bond that pays proportionally to the UK RPI in 10 years, we know that at maturity we will be able to buy a certain number of "baskets" of goods and services.

There is a long and varied history of the government issuance of index-linked bonds [1]. We will just look at a few of the main government issues currently outstanding, as these are the main providers of liquidity in the inflation market. The UK government started issuing gilts linked to the retail price index (RPI) in 1981, and there are currently eleven different issues outstanding with an inflation-uplifted value of 109 billion GBP and with maturities ranging from three to fifty years. Note that this constitutes approximately a quarter of the UK government's net debt.

The UK index-linked gilts are "capital-indexed" bonds, which means that both the coupon payments and the final redemption are proportional to the price index. There are semi-annual coupon payments of the amount:

$$\text{payment}(T_i) = \frac{c}{2} \times N \times \frac{I(T_i - \tau_{\text{delay}})}{I(T_{\text{ref}})}. \tag{1}$$

where c is the coupon (between 1% and 4%), N is the notional, and τ_{delay} is the time lag (originally 8 months, but 3 months and interpolated for issues since 2005). The redemption payment at the final time T_{f} is

$$\text{payment}(T_{\text{f}}) = N \times \frac{I(T_{\text{f}} - \tau_{\text{delay}})}{I(T_{\text{ref}})}. \tag{2}$$

The US government started issuing treasury notes linked to the CPI-U in 1997. These are now known as Treasury Inflation-Protected Securities (TIPS) and there is currently over 300 billion USD outstanding with maturities up to thirty years. The structure of the bonds is broadly similar to those of the UK. An important difference is that the final redemption payment is floored to the original nominal value. This was a useful device to attract investors: they can't lose money even under deflation, and costs very little to the government as the possibility of the floor being activated is so remote.

The French government started issuing bonds linked to the FCPI-x in 1998,[6] and then also to the MUICP-x in 2001. There is approximately 50 billion EUR outstanding of these "OAT-i"s with maturities up to thirty years. Finally, after a constitutional ban on indexation since 1949, the German government launched its first index-linked bond, with a ten-year maturity, in March 2006, so that now all of the G7 governments have some form of index-linked bonds outstanding. Interestingly, the choice of price index was the MUICP-x, rather than a German price index.

There are several reasons that a government might wish to have part of its debt linked to inflation. An important factor is that its income (i.e., revenues from taxation) is likely to always rise with inflation, hence there is a matching of liability to future revenues. In addition, the government would clearly hope to obtain cheaper borrowing through index-linking due to an "inflation-risk premium". Other reasons include the enhancement of credibility of economic policy regarding inflation (putting its money where its mouth is), and also the importance of making inflation-linked products available to those with future inflation-linked liabilities (most importantly pension funds). Finally, the development of a liquid market in index-linked bonds allows the government, the monetary-policy decision makers, and others to observe the market's view on future levels of inflation.

[6] The exclusion of tobacco from the index is due to an older piece of French legislation that forbade linking of securities to tobacco prices [1]. Once established, there is a resistance to changing the choice of index in later issues due to fears of increasing complexity and decreasing liquiudity. This is also seen in the continued choice of the RPI for UK index-linked gilts, though now the UK monetary policy is directed towards the CPI (which excludes mortgage payments, council tax and other things).

The main reason an institution would buy an inflation-linked bond is to protect their own inflation-linked exposure. For example, most pensions increase from year to year with inflation, and there are sometimes statutory requirements for this. Therefore a pension fund or a life-insurer can match this future liability by buying these bonds. Another advantage of index-linked bonds when compared to conventional riskless bonds whose real value declines with time is that the effective duration of an index-linked bond is much longer, which is useful for pension funds and others who want guaranteed cashflows in forty years time. Finally many individual investors are attracted to the guarantee of real returns from inflation-linked bonds. In the UK it is straightforward buy inflation protection at a post office branch through an index-linked savings certificate, and similar government-backed retail products exist in other countries.

Relatively recently there has been a growing corporate issuance of index-linked bonds. In principle this is in any company's interests that has income related to the general level of prices, e.g., a supermarket chain. However, the main issuers in the UK tend to be utility companies, who are restricted by regulators to increasing prices in line with inflation. The current market capitalization of UK non-government index-linked bonds is approximately 12 billion pounds, not much more than a tenth of the government issuance. Of Euro non-government index-linked bonds it is even smaller at approximately 11 billion EUR.

1.2.1 Zero-coupon Inflation Bond

In analogy to the standard use of a zero-coupon discount bond when considering the term structure of interest rates, we can introduce a zero-coupon inflation bond that pays the value of a price index $I(T)$ at maturity T. While such bonds have been issued, e.g., in Sweden, they are generally not available directly, but are a useful theoretical tool. At an earlier time t the idealized zero-coupon bond has an arbitrage-free price of $P_{\mathrm{I}}(t,T)$, with no delay from the index date to payment. The value of the payments of different inflation-linked bonds can then be written in terms of $P_{\mathrm{I}}(t,T)$, for instance the payment in (1) has the value at time $t < T_i$ of,

$$v(t) = \frac{Nc}{2} \frac{P_{\mathrm{I}}(t, T_i - \tau_{\mathrm{delay}})}{I(T_{\mathrm{ref}})}. \tag{3}$$

In principle, then, the present value of index-linked bonds in the market can be used to extract at least some points on the curve of $P_{\mathrm{I}}(t,T)$. Unfortunately, as there is a rather limited supply of index-linked bonds of different maturities, this is not very practical. At present the best way to extract the market value of $P_{\mathrm{I}}(t,T)$ for the Eurozone, UK, or USA is from the prices of inflation swaps, which we consider in the next section.[7]

1.2.2 Market-projected price index

Following from the introduction above of the zero-coupon index-linked bond we can define the "forward" or "projected" value of the price index at a later time T by,

$$\hat{I}(t,T) = \frac{P_{\mathrm{I}}(t,T)}{P(t,T)}, \tag{4}$$

where $P(t,T)$ is the value of a zero-coupon discount bond with maturity T. This will prove to be a useful combination when calculating index-linked derivatives.

[7] Interestingly, the value of $P_{\mathrm{I}}(t,T)$ at long maturities extracted from UK inflation swaps is slightly higher than that from index-linked gilts[2]. For the US the discrepancy exists at all maturities, while for the Eurozone, the index-linked swaps and bonds are more consistent. The fact that such differences are not removed by arbitrage, especially in the US, shows that the index-linked markets are still relatively illiquid.

2 Inflation-Linked Derivatives

The trading of products linked to inflation other than index-linked bonds has grown in the UK and the Eurozone since the late 1990s and more recently in the USA. An index-linked swap can be tailor-made to match the requirements of a client (future liabililities) more closely than is possible from the limited range of index-linked bonds. At present, these products are traded over-the-counter. As an example, a pension fund can enter an inflation swap agreement with an investment bank which together with the conventional bonds it holds allows the synthesization of an index-linked bond. Over 50 billion GBP notional value of UK and Eurozone inflation swaps were traded in 2005.[8]

2.1 Inflation Swaps

The index-linked derivatives market is dominated by zero-coupon inflation swaps, which are becoming more easily available. For instance, RBS now offers electronic trading of inflation swaps (UKRPI with maturities from two to fifty years, FCPI-x and MUICP-x with maturities from one to thirty years).

A zero-coupon swap consists of agreeing to pay the fixed amount in T years of $[(1+K)^{T-T_0}-1]$ in exchange for receiving the relative increase of an index, $[I(T)-I(T_0)]/I(T_0)$. This swap has zero value if,

$$(1+K)^{T-T_0} = \frac{\hat{I}(t,T)}{I(T_0)}. \tag{5}$$

It is the fixed rate K that is usually quoted, e.g., on Bloomberg.

An alternative swap which is an important building block of more exotic derivatives is the year-on-year swap. This contract agrees to pay K at the time T and receive the average rate of inflation over the previous year, $[I(T)/I(T-1)]-1$. It is not possible, however, to write the present value of the payment of $I(T)/I(T-1)$ simply in terms of zero-coupon inflation bonds. This relatively simple instrument therefore requires a choice of model to be valued (see section 3.3.3) and this could be why it is traded at much lower volumes than zero-coupon swaps.

2.2 Inflation Futures

While inflation derivatives are mainly traded over the counter, there have been various attempts to list inflation-related products on exchanges. For example, futures contracts paying one million EUR times the year-on-year inflation rate from the Eurozone HICP are now traded on the Chicago Mercantile Exchange (CME). The available maturities run monthly, but only up to a year ahead, so these are in effect related to zero-coupon inflation bonds/swaps.

2.3 Inflation Options

A standalone inflation call or put can be defined in analogy to calls and puts on more conventional assets. Generally the market defines the payoff of an inflation option in terms of the relative change of the price index from a reference date to the expiry date compared to the level given by a fixed rate, i.e., the payoff of an inflation call is,

$$\text{payoff} = \left(\frac{I(T)}{I(T_0)} - (1+K)^{(T-T_0)}\right)_+ \tag{6}$$

(Note this is not quite a bona-fide call option as $I(t)$ is not a tradable asset. However, hedging strategies can be constructed if there is a liquid market in index-linked bonds

[8] ICAP estimate

and swaps). An option will be "at the money" when the strike is close to $K = -1 + (\hat{I}(t,T)/I(T_0))^{1/(T-T_0)}$. Clearly, the price of an inflation option is model dependent, and will largely be determined by an estimate of the inflation "volatility". Simple models based on no-arbitrage arguments that allow such options to be priced are presented in section 3.

A more commonly traded class of inflation option, especially for Eurozone inflation, is a cap or floor on the year-on-year inflation rate. This product is a series of payments, usually annual, which are given by (for a caplet):

$$\text{Payoff}_i = \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - (T_i - T_{i-1})K \right)_+ . \tag{7}$$

A common source of optionality in the market is embedded in the structure of some index-linked bonds. For instance there is a certain (very small) value to the deflation floors in the redemption payment of US and French government-issued bonds. Commercially issued European inflation bonds tend to have coupons related to the year-on-year inflation, and floored at 0%. A more complex product is popular in the UK: Many corporations have issued Limited Price Indexed (LPI) bonds. The LPI is defined as an index that increases every year by the RPI, except if the RPI has gone down then the LPI stays fixed at the previous year's value, while if the RPI goes up by more than 5% then the LPI only increases by 5%. This structure is related to UK pension legislation which requires some pension payments to increase with the LPI. It is also considerably harder to model than a year-on-year swap with cap/floor (see section 3.3.7)

2.4 Price visibility

Reflecting their dominant volume, it is easiest to find market prices of zero-coupon swaps, e.g., swap rates are quoted on Bloomberg. Monthly futures prices up to a year can be found on the web site of the CME. For options on inflation one generally has to rely on quotes from brokers, e.g., from Tullets or ICAP. For the Eurozone, the price of year-on-year swaps with a few different caps and floors (mainly 0% floors) are quoted. For the UK, the most available option quote is in the form of LPI bonds. These are quoted in the form of the spread in projected yield between an LPI zero-coupon bond and an RPI zero-coupon bond.

3 Inflation models for derivatives pricing

The modelling of inflation derivatives pricing has developed recently, and there are only a few relevant papers to be found in the literature. The most quoted model is the Jarrow-Yildirim (JY) model published in 2003 [4]. We briefly review this model in section 3.1 and then an alternative "market model" in section 3.2. Then we introduce a relatively simple two-process Hull-White model for inflation in section 3.3 and use it to find prices for simple inflation options.

3.1 Jarrow-Yildirim Model

This model assumes that in addition to a conventional money market there is a separate market for real returns. The price index then gives us a conversion factor between these two economies. By analogy to two-currency interest rate model, it is then proposed that the real and nominal returns are determined by instantaneous forward rates that follow the (Hull-White) processes:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \tag{8}$$

$$df_{\mathrm{R}}(t,T) = \alpha_{\mathrm{R}}(t,T)dt + \sigma_{\mathrm{R}}(t,T)dW_{\mathrm{R}}(t). \tag{9}$$

Meanwhile, the price index $I(t)$ is taken to follow a log-normal process:

$$\frac{dI}{I} = \mu_{\rm I}(t)dt + \sigma_{\rm I} dW_{\rm I}(t) \tag{10}$$

The Brownian motions $W(t)$, $W_{\rm R}(t)$ and $W_{\rm I}(t)$ are all correlated which will require three correlation parameters to be defined.

As an index-linked bond guarantees a real return, we can interpret this as the discount bond in the real economy valued in the nominal currency, i.e., we have $P_{\rm I}(t,T) = I(t)P_{\rm R}(t,T)$. Standard no-arbitrage arguments tell us that if we can find a change of measure where the combinations $P(t,T)/B(t)$, $I(t)P_{\rm R}(t,T)/B(t)$ and $I(t)B_{\rm R}(t)/B(t)$ are all martingales, then the price of a given security with payoff $Q(T)$ is given by the expectation in this measure,

$$v_Q(t) = B(t)\mathbb{E}\left[Q(T)B^{-1}(T)\right]. \tag{11}$$

Here $B(t) = \exp[-\int_0^t r(t')dt']$ is a continuously compounding cash account, with $r(t) = f(t,t)$ the instantaneous short rate. For instance, the zero-coupon index bond will have the value,

$$P_{\rm I}(t,T) = \mathbb{E}\left[I(T)e^{-\int_t^T r(t')dt'}\right]. \tag{12}$$

Note that, while the terms entering the expectation do not include the real forward rates anywhere, they will enter into the dynamics of $I(T)$ under the risk-neutral measure[4].

For more explicit calculations, JY chose the volatility structure,[9] $\sigma(t,T) = \sigma e^{-a(T-t)}$ and $\sigma_{\rm R}(t,T) = \sigma_{\rm R} e^{-a_{\rm R}(T-t)}$, which is well-known to be equivalent to the Hull-White short-rate model[6]:

$$dr = a[\theta(t) - r(t)]dt + \sigma dW, \tag{13}$$

$$dr_{\rm R} = a_{\rm R}[\theta_{\rm R}(t) - r_{\rm R}(t)]dt + \sigma_{\rm R} dW_{\rm R}, \tag{14}$$

and JY used historical data to estimate the value of the constant parameters.

3.2 Market Model

A different model inspired by the successful interest rate market models has been suggested where the market projected or forward price index $\hat{I}(t,T)$ (see section 1.2.2) is modelled as a log-normal process [3, 5]. In this model, one first chooses the discrete set of times relevant to the product (eg. a floored swap with annual payments). One then proposes a separate dynamics for the forward index for each of these times:

$$\frac{d\hat{I}(t,T_i)}{\hat{I}(t,T_i)} = (\ldots)dt + \sigma_{{\rm I},i} dW_{{\rm I},i} \tag{15}$$

(we do not specify a drift, as this will be fixed in the risk-neutral measure). The stochastic interest rates then proceed within a conventional market model,

$$\frac{df(t,T_{i-1},T_i)}{f(t,T_{i-1},T_i)} = (\ldots)dt + \sigma_i dW_i. \tag{16}$$

The convenience of this model is that, by choosing $P(t,T_i)$ as a numeraire, the combination $\hat{I}(t,T) = P_{\rm I}(t,T)/P(t,T)$ should be a martingale in the risk-neutral measure, hence the dynamics in this measure obey

$$\frac{d\hat{I}(t,T_i)}{\hat{I}(t,T_i)} = \sigma_{{\rm I},i} dW_{{\rm I},i}, \tag{17}$$

[9] In the risk-neutral measure, $\alpha(t,T)$ is fixed by the choice of $\sigma(t,T)$.

and we also have,

$$\frac{df(t,T_{i-1},T_i)}{f(t,T_{i-1},T_i)} = \sigma_i dW_i, \tag{18}$$

with no drift terms. A product that depends on the index at time T_i will have an arbitrage-free price at time t of,

$$\begin{aligned} v(t) &= P(t,T_i)\mathbb{E}\left[Q[I(T_i)]\right] \\ &= P(t,T_i)\mathbb{E}\left[Q[\hat{I}(T_i,T_i)]\right]. \end{aligned} \tag{19}$$

The difficulty comes when a product also needs the value of the index at a different time, for instance in a year-on-year inflation swap. In the T_i forward measure, the dynamics of $\hat{I}(t,T_{i-1})$ will have a drift term. However, this is straightforward to handle [5], and leads to a simple result for one leg of a year-on-year swap which depends on three volatility parameters and two correlation parameters.

3.3 Two-process Hull-White model

An alternative way of modelling inflation is suggested by the fact that the two quantities that enter expectations (if the cash bond is numeraire) are the short interest rate and the level of the price index at different times. We could therefore only include these two processes in our model. No-arbitrage coniditions can be found by making sure that we use a change of measure where $P(t,T)/B(t)$ and $P_{\rm I}(t,T)/B(t)$ are both martingales. In this way we ignore the existence of a "real" economy.

In this way of thinking, we regard the price index as an asset, which we would like to model realistically. It is not unreasonable, however, to consider the *rate* of inflation as the driving stochastic process, and the price index is an integral over this rate,

$$I(T) = I(T_0)e^{\int_{T_0}^{T} i(t')dt'}. \tag{20}$$

Mean reversion is to be expected of this rate, and so due to its tractability we choose a Hull-White model for the inflation rate:

$$di = \alpha_{\rm I}[\theta_{\rm I}(t) - i(t)]dt + \sigma_{\rm I} dW_{\rm I}. \tag{21}$$

The mean reversion level, $\theta_{\rm I}(t)$, is allowed to be time dependent so as to include any change of measure. The correct dependence in the risk-neutral measure will be found below.

It is natural (though not obligatory) with this choice of model for inflation to also model interest rates with a Hull-White model,

$$dr = \alpha[\theta(t) - r(t)]dt + \sigma dW, \tag{22}$$

and the driving Brownian motions in the two processes are correlated, $\overline{dW(t)dW_{\rm I}(t')} = \rho\delta(t-t')dtdt'$ (in principle we could lag this correlation).

Disadvantages of the two-process Hull-White model for inflation include the fact that a strong smile can not be captured in the price of inflation caps/floors versus the strike (this is also true for the JY and market models). Also, there is a well-known problem of the Hull-White model for interest rates: The maturity dependence of implied Black volatilities from market prices of caplets usually has a "hump", but the Hull-White model always gives a monotonically decreasing volatility with maturity. (In contrast, the term structure of inflation caps/floors seems to be well captured by the Hull-White model, see later.) Finally, any product that depends on the correlation of different forward interest rates will not be well described by this model.

3.3.1 Risk-neutral measure

The basic market-observables are taken to be the zero-coupon discount bond, which has the value,

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(t')dt'} \middle| r(t) \right], \tag{23}$$

and the zero-coupon index-linked bond, which is given by,

$$P_{\rm I}(t,T) = I(t)\mathbb{E}\left[e^{\int_t^T [i(t')-r(t')]dt'} \middle| i(t), r(t) \right]. \tag{24}$$

Standard arguments (see e.g., [7]) show that the risk neutral measure (when the cash account is numeraire) is given when the mean reversion level of the short interest rate is,

$$\theta(t) = f(0,t) + \frac{1}{\alpha}\frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{2\alpha^2}\left(1 - e^{-2\alpha t}\right), \tag{25}$$

where we have defined the instantaneous forward rate,

$$f(t,T) = -\frac{\partial}{\partial T}\ln P(t,T) \tag{26}$$

and $P(0,t)$ are the observed market discount bonds. Note that the initial short rate is given in the market by $r(0) = f(0,0)$.

In an analogous fashion we find that the mean reversion level of inflation in the risk neutral measure is,

$$\begin{aligned} \theta_{\rm I}(t) &= f_{\rm I}(0,t) + \frac{1}{\alpha}\frac{\partial}{\partial t} f_{\rm I}(0,t) - \frac{\sigma_{\rm I}^2}{2\alpha_{\rm I}^2}\left(1 - e^{-2\alpha_{\rm I} t}\right) \\ &+ \frac{\rho\sigma\sigma_{\rm I}}{\alpha_{\rm I}}\left[\frac{1}{\alpha}(1-e^{-\alpha t}) + \frac{1}{\alpha_{\rm I}}(1-e^{-\alpha_{\rm I} t})e^{-\alpha t}\right]. \end{aligned} \tag{27}$$

Here we have defined the forward inflation rate as,

$$f_{\rm I}(t,T) = \frac{\partial}{\partial T}\ln \hat{I}(t,T), \tag{28}$$

with $\hat{I}(t,T) = P_{\rm I}(t,T)/P(t,T)$ known as the projected price index.

3.3.2 Closed-form solution

The inflation rate evolves as:

$$i(t) = i(s)e^{-\alpha_{\rm I}(t-s)} + \alpha_{\rm I}\int_s^t dt'\theta_{\rm I}(t')e^{-\alpha_{\rm I}(t-t')} + \sigma_{\rm I}\int_s^t dW_{\rm I}(t')e^{-\alpha_{\rm I}(t-t')} \tag{29}$$

and the short interest rate has the evolution:

$$r(t) = r(s)e^{-\alpha(t-s)} + \alpha\int_s^t dt'\theta(t')e^{-\alpha(t-t')} + \sigma\int_s^t dW(t')e^{-\alpha(t-t')}. \tag{30}$$

Clearly these are both normally distributed. Similarly the integrals over these rates are normal distributions: the integral between T_1 and T_2 given the value of $i(t)$ where $t \le T_1 < T_2$ is:

$$\begin{aligned} \int_{T_1}^{T_2} i(t')dt' &= i(t)e^{-\alpha_{\rm I}(T_1-t)}B(\alpha_{\rm I}, T_2-T_1) \\ &+ \alpha_{\rm I}\int_t^{T_1} dt'\theta_{\rm I}(t')e^{-\alpha_{\rm I}(T_1-t')}B(\alpha_{\rm I}, T_2-T_1) + \alpha_{\rm I}\int_{T_1}^{T_2} dt'\theta_{\rm I}(t')B(\alpha_{\rm I}, T_2-t') \\ &+ \sigma_{\rm I}\int_t^{T_1} dW_{\rm I}(t')e^{-\alpha_{\rm I}(T_1-t')}B(\alpha_{\rm I}, T_2-T_1) + \sigma_{\rm I}\int_{T_1}^{T_2} dW_{\rm I}(t')B(\alpha_{\rm I}, T_2-t') \end{aligned} \tag{31}$$

and similarly the integral over the short interest rate, given the value $r(t)$ is

$$\begin{aligned}\int_{T_1}^{T_2} r(t')dt' &= r(t)e^{-\alpha(T_1-t)}B(\alpha,T_2-T_1) \\ &\quad +\alpha\int_t^{T_1} dt'\theta(t')e^{-\alpha(T_1-t')}B(\alpha,T_2-T_1)+\alpha\int_{T_1}^{T_2} dt'\theta(t')B(\alpha,T_2-t') \\ &\quad +\sigma\int_t^{T_1} dW(t')e^{-\alpha(T_1-t')}B(\alpha,T_2-T_1)+\sigma\int_{T_1}^{T_2} dW(t')B(\alpha,T_2-t').\end{aligned} \tag{32}$$

Here we have used the definition $B(\alpha,\tau)=(1-e^{-\alpha\tau})/\alpha$, as this combination crops up frequently in the Hull-White model. It is then straightforward, if sometimes tedious, to find the means and variances of the various combinations of integrals over rates that appear in different products.

3.3.3 Year-on-year inflation swap

This product was described in section 2.1. Here we value at time t the payoff at time T_2 of $\frac{I(T_2)}{I(T_1)}$, with $t<T_1<T_2$. Within a two-process short-rate model this has the value,

$$v_{\mathrm{YoY}}(t,T_1,T_2)=\mathbb{E}\left[e^{-\int_t^{T_2} r(t')dt'}e^{\int_{T_1}^{T_2} i(t')dt'}\right]. \tag{33}$$

Now, if the integrals over r and i are normal distributions, as they are in the Hull-White model, then we can use the result

$$\mathbb{E}\left[e^{-\int_t^{T_2} r(t')dt'+\int_{T_1}^{T_2} i(t')dt'}\right]=\frac{\mathbb{E}\left[e^{\int_t^{T_2}[i(t')-r(t')dt'}\right]\mathbb{E}\left[e^{-\int_t^{T_1} r(t')dt'}\right]}{\mathbb{E}\left[e^{\int_t^{T_1}[i(t')-r(t')]dt'}\right]}e^{\mathrm{Covar}[\int_t^{T_1} i,\int_{T_1}^{T_2}(r-i)]}, \tag{34}$$

to find that

$$v_{\mathrm{YoY}}(t,T_1,T_2)=P(t,T_1)\frac{P_{\mathrm{I}}(t,T_2)}{P_{\mathrm{I}}(t,T_1)}e^{C(t,T_1,T_2)}, \tag{35}$$

where C is a correction due to correlations between integrals over different intervals:

$$C(t,T_1,T_2)=\mathrm{Covar}\left[\int_t^{T_1} i,\int_{T_1}^{T_2}(r-i)\right]. \tag{36}$$

Using the results for the integrals on a given path, (31) and (32), we find that in the two-process Hull-White model the correction factor is given by

$$\begin{aligned}C(t,T_1,T_2) &= -\frac{\sigma_{\mathrm{I}}^2}{2}B^2(\alpha_{\mathrm{I}},T_1-t)B(\alpha_{\mathrm{I}},T_2-T_1) \\ &\quad -\frac{\rho\sigma_{\mathrm{I}}\sigma}{\alpha_{\mathrm{I}}}B(\alpha,T_2-T_1)[B(\alpha_{\mathrm{I}}+\alpha,T_1-t)-B(\alpha,T_1-t)].\end{aligned} \tag{37}$$

As the correlation parameter tends to be positive, we see that this covariance is negative, and the "convexity correction" reduces the value of the year-on-year swap.

3.3.4 Inflation Options

Inflation options are defined in section 2.3. Within a two Hull-White process model an inflation call has a price of

$$v_{\mathrm{Icall}}(t,T,X)=\mathbb{E}\left[e^{-\int_t^T r(t')dt'}\left(\frac{I(t)}{I(T_0)}e^{\int_t^T i(t')dt'}-X\right)_+\right] \tag{38}$$

where $I(t)$ is the most recent published value of the index at time t, and $X = (1+K)^{(T-T_0)}$. As this is of the general form $E[e^y(ae^x - b)]$ with x and y correlated gaussian-distributed variates, the expectation can be evaluated, and is of the "Black-Scholes" form:

$$v_{\text{Icall}} = \left\{ \frac{P_{\text{I}}(t,T)}{I(T_0)} \Phi \left[\frac{\ln\left(\frac{P_{\text{I}}(t,T)}{I(T_0)P(t,T)X}\right) + \frac{V_{\text{I}}(T-t)}{2}}{\sqrt{V_{\text{I}}(T-t)}} \right] - XP(t,T)\Phi \left[\frac{\ln\left(\frac{P_{\text{I}}(t,T)}{I(T_0)P(t,T)X}\right) - \frac{V_{\text{I}}(T-t)}{2}}{\sqrt{V_{\text{I}}(T-t)}} \right] \right\}, \tag{39}$$

where the variance, V_{I}, is given by,

$$V_{\text{I}}(\tau) = \text{Var}\left[\int_t^{t+\tau} i(t)dt\right] = \frac{\sigma_{\text{I}}^2}{\alpha_{\text{I}}^2}\left[\tau - B(\alpha_{\text{I}},\tau) - \frac{\alpha_{\text{I}}}{2}B(\alpha_{\text{I}},\tau)^2\right]. \tag{40}$$

We can write this result as a Black formula:

$$v_{\text{Icall}}(t,T,X) = v_{\text{Black-call}}\left(\frac{\hat{I}(t,T)}{I(T_0)}, X, \sqrt{V_{\text{I}}(T-t)}; T\right), \tag{41}$$

where $v_{\text{Black-call}}(f, X, \sigma\sqrt{T}; T)$ is the standard Black result for a call on a contract at time T with strike K, forward price f, and volatility σ,

$$v_{\text{Black-call}}(f, X, y; T) = P(t,T)[f\Phi(d_1) - X\Phi(d_2)] \tag{42}$$

with $d_{1,2} = [\ln(f/x) \pm y^2/2]/y$.

It is straightforward to show that an inflation put has the price,

$$v_{\text{Iput}}(t,T,X) = v_{\text{Black-put}}\left(\frac{\hat{I}(t,T)}{I(T_0)}, X, \sqrt{V_{\text{I}}(T-t)}; T\right), \tag{43}$$

with

$$v_{\text{Black-put}}(f, X, y; T) = P(t,T)[-f\Phi(-d_1) + X\Phi(-d_2)]. \tag{44}$$

Note that the long-term variance (for maturities $T - t \gg 1/\alpha_{\text{I}}$) has the form,

$$V_{\text{I}} \approx \frac{\sigma_{\text{I}}^2}{\alpha_{\text{I}}^2}(T-t), \tag{45}$$

which makes the final result look like an option on a log-normal process with volatility equal to $\sigma_{\text{I}}/\alpha_{\text{I}}$.

3.3.5 Inflation Caps and Floors

The payoffs in a year-on-year cap were defined in section 2.3. This has the same appearance of the reciprocal of a future price index as the year-on-year swap, so it is not surprising that the present value of the ith payment has a Black-Scholes form, but with a convexity correction on the forward:

$$\begin{aligned} v_{\text{Icaplet}} &= \mathbb{E}\left[e^{-\int_t^{T_2} r(t')dt'}\left(\frac{I(T_2)}{I(T_1)} - 1 - K\tau\right)_+\right] \\ &= \mathbb{E}\left[e^{-\int_t^{T_2} r(t')dt'}\left(e^{\int_{T_1}^{T_2} i(t')dt'} - (1+K\tau)\right)_+\right] \\ &= P(t,T_2)\left[\frac{\hat{I}(t,T_2)e^{C(t,T_1,T_2)}}{\hat{I}(t,T_1)}\Phi(d_+) - (1+K\tau)\Phi(d_-)\right], \end{aligned} \tag{46}$$

with

$$d_{\pm} = \frac{1}{\sqrt{V_{\rm I}(t,T_1,T_2)}}\left[\ln\left(\frac{\hat{I}(t,T_2)e^{C(t,T_1,T_2)}}{\hat{I}(t,T_1)(1+K\tau)}\right) + \frac{V_{\rm I}(t,T_1,T_2)}{2}\right] \tag{47}$$

The variance here is given by,

$$\begin{aligned} V_{\rm I}(t,T_1,T_2) &= {\rm Var}\left[\int_{T_1}^{T_2} i(t')dt'|i(t)\right] = \frac{\sigma_{\rm I}^2}{\alpha_{\rm I}^2}\left[T_2 - T_1 - B(\alpha_{\rm I},T_2-T_1) - \frac{\alpha_{\rm I}}{2}B(\alpha_{\rm I},T_2-T_1)^2\right] \\ &\quad + \sigma_{\rm I}^2 B(2\alpha_{\rm I},T_1-t)B(\alpha_{\rm I},T_2-T_1)^2. \end{aligned} \tag{48}$$

We can write this result as a Black formula:

$$v_{\rm Icaplet}(t,T_1,T_2,K) = v_{\rm Black-call}\left(\frac{\hat{I}(t,T_2)e^{C(t,T_1,T_2)}}{\hat{I}(t,T_1)}, (1+K\tau), \sqrt{V_{\rm I}(t,T_1,T_2)}; T_2\right). \tag{49}$$

and a similar analysis gives the result for an inflation floorlet:

$$v_{\rm Ifloorlet}(t,T_1,T_2,K) = v_{\rm Black-put}\left(\frac{\hat{I}(t,T_2)e^{C(t,T_1,T_2)}}{\hat{I}(t,T_1)}, (1+K\tau), \sqrt{V_{\rm I}(t,T_1,T_2)}; T_2\right). \tag{50}$$

3.3.6 Delayed payments

Typically the payment date on an index-linked derivative is later than the date for the price index by two to three months. This entails another correction for correlations. For example, with a year-on-year swap which is paid at $T_{\rm pay} > T_2$ we have

$$\begin{aligned} v_{\rm YoY}(t,T_1,T_2,T_{\rm pay}) &= \mathbb{E}\left[e^{-\int_t^{T_{\rm pay}} r(t')dt'} e^{\int_{T_1}^{T_2} i(t')dt'}\right] & (51) \\ &= P(t,T_1)\frac{P(t,T_{\rm pay})}{P(t,T_2)}\frac{P_{\rm I}(t,T_2)}{P_{\rm I}(t,T_1)}e^{C(t,T_1,T_2)}e^{C_2(t,T_1,T_2,T_{\rm pay})} & (52) \end{aligned}$$

with $C_2 = {\rm Covar}\left[\int_{T_2}^{T_{\rm pay}} r, \int_{T_1}^{T_2} i\right]$. Calculating the covariance gives:

$$C_2(t,T_1,T_2,T_{\rm pay}) = \frac{\rho\sigma_{\rm I}\sigma}{\alpha_{\rm I}}B(\alpha,T_{\rm pay}-T_2)\left[B(\alpha_{\rm I}+\alpha,T_2-t) - B(\alpha,T_2-T_1) - e^{-\alpha(T_2-T_1)}B(\alpha_{\rm I}+\alpha,T_1-t)\right] \tag{53}$$

3.3.7 Limited Price Indexed bond

In the UK, many bonds and swaps are related to the LPI. This index is a capped and floored version of the RPI (usually 0% to 5%). The LPI is typically only defined on one date of every year, such that it increases by the same relative amount as the RPI over the previous year, unless it is capped or floored. That is:

$$I_{\rm LPI}(T_i) = I_{\rm LPI}(T_{i-1}){\rm mid}\left(1+K_{\rm floor}, \frac{I(T_i)}{I(T_{i-1})}, 1+K_{\rm cap}\right), \tag{54}$$

where ${\rm mid}(x,y,z) = \max(x,\min(y,z))$. These are difficult to price in terms of zero-coupon index bonds based on the RPI, because of the compounding feature in every year's cap and floor.

We can rewrite the LPI as a product:

$$I_{\rm LPI}(T_m) = I_{\rm LPI}(T_0)\prod_{n=1}^{m}\left\{1+K_{\rm floor} + \left[\frac{I(T_n)}{I(T_{n-1})} - 1 - K_{\rm floor}\right]_+ - \left[\frac{I(T_n)}{I(T_{n-1})} - 1 - K_{\rm cap}\right]_+\right\}. \tag{55}$$

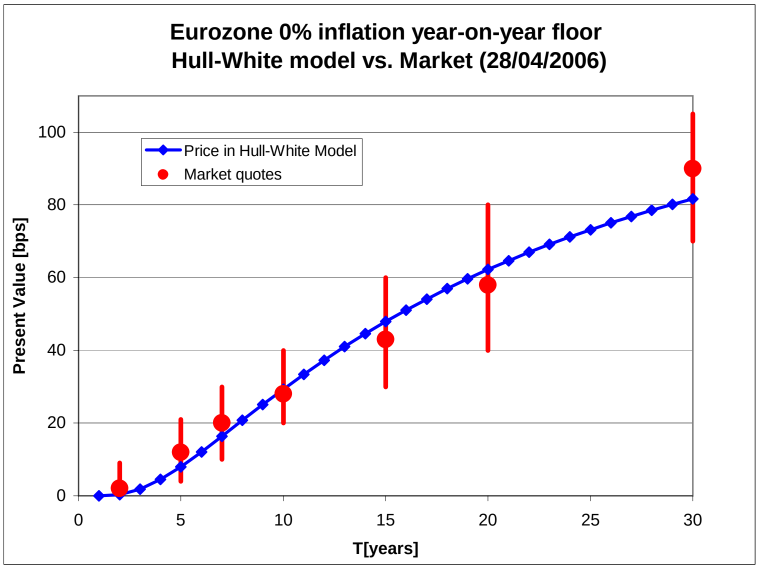

Figure 2: Prices for a zero-percent year-on-year inflation floor for the Eurozone index MUICP-x. The red circles are the midpoints of market quotes (from broker, April 2006) with red lines showing bid-offer spreads. The blue diamonds are prices found with the Hull-White model using (50) with $\alpha = 0.1$, $\sigma = 1\%$, $\alpha_\mathrm{I} = 0.15$, $\sigma_\mathrm{I} = 0.93\%$ and $\rho = 40\%$. While this is quite a good fit as a function of maturity of the product, it is not possible to get consistent prices from this model to the market across a range of strikes.

Therefore, the price of a zero-coupon LPI bond can be written as the discounted expectation of this,

$$P_{\mathrm{LPI}}(t,T) = \mathbb{E}\left[I_{\mathrm{LPI}}(T)e^{-\int_t^T r(t')dt'}\right] \tag{56}$$

which we can do with a Monte Carlo calculation. Approximate ways of calculating this can be found, for instance by ignoring correlations between the different terms in the product, but they are of restricted validity.

3.3.8 Calibration

The simplest part of calibrating the two-process Hull-White inflation model is the mean reversion level, which is directly found from the market curves for $P_\mathrm{I}(t,T)$ and $P(t,T)$. In practice it is harder to calibrate the volatility and reversion speed parameters, σ_I and α_I, and the correlation ρ, because of the relative scarcity of available market prices for inflation options (we assume that the interest-rate Hull-White model has already been calibrated to pure interest-rate products).

In figure 2 we show our calibration to Eurozone inflation floors at 0%. The maturity dependence is well captured by the inflation Hull-White model. While these are the most traded products, market prices are available at other strikes, and we need a different volatility in our model to match these. This "smile" in implied inflation volatility is shown in figure 3, where we plot the values of σ_I needed to match market quotes of inflation options at different strikes.

In figure 4 we show our calibration to broker quotes of the LPI-vs-RPI spread with a range of caps and floors. The model results here are found with Monte Carlo sampling. While this is quite satisfactory, it should be pointed out that the value of the cap tends to cancel with that of the floor, and so there is not as much sensitivity to parameters as there would be with just a floor or a cap.

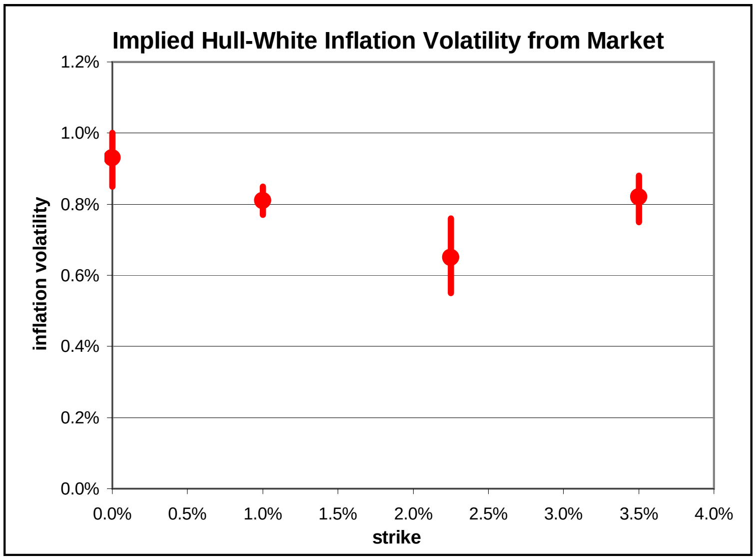

Figure 3: The values of the model parameter $\sigma_{\rm I}$ needed to match market quotes for options on Eurozone inflation (ten-year maturity, source: ICAP) with all other model parameters fixed. Error bars are bid-offer spreads. The strike at 2.25% is close to at-the-money (i.e., reflects market expectation of year-on-year inflation rate). This shows the strong smile, with options for strikes away from at-the-money more expensive relatively than the inflation Hull-White model predicts.

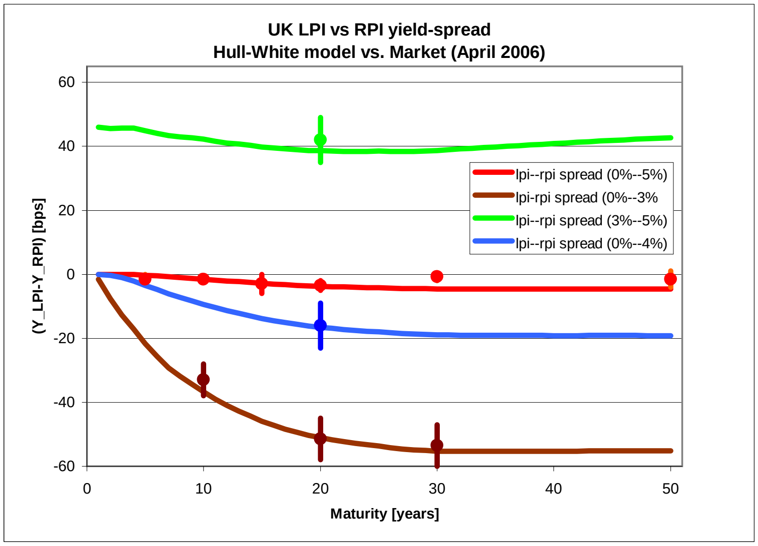

Figure 4: Yield spreads between a zero-coupon LPI bond and a zero-coupon RPI bond. The different curves represent different ranges of cap and floor for the LPI. The circles are the midpoints of market quotes (from broker, April 2006) with bars showing bid-offer spreads. The curves are prices found with the Hull-White model using Monte Carlo sampling, with $\alpha = 0.1$, $\sigma = 1\%$, $\alpha_{\rm I} = 0.1$, $\sigma_{\rm I} = 0.6\%$ and $\rho = 40\%$. The fit is good over maturity *and* over different strikes. However, these prices are not as sensitive as a pure floor or cap, making calibration easier, but less reliable.

4 Monte Carlo pricing with local volatility of inflation

The fact that we cannot recover the market prices of Eurozone inflation floors of 0% and caps/floors closer to the money is an important drawback of the two-process Hull-White model. Of the various alternative models that might give this "smile" (e.g., jumps, stochastic volatility), we have set up a local-volatility version of the Hull-White model. Analytical solutions will no longer be possible but Monte Carlo sampling within a short-stepping framework remains much the same as for the original model. In general, local volatility models are an efficient way to match to option prices, and give good prices for more exotic derivatives such as barrier options. However, they do not handle well those derivatives with true dynamic features such as options with forward starts. To our knowledge such options are not actively traded on inflation at present.

We start with a general "short-rate" process,

$$dx = -\alpha_{\rm I}(x)xdt + \sigma_{\rm I}(x)dW_{\rm I}. \tag{57}$$

We then write the inflation rate as $i(t) = x(t) + \varphi(t)$, where $\varphi(t)$ is a function that will determine the change of measure we need to make $P_{\rm I}(t,T)/B(t)$ a martingale (see [7] for an explanation of this procedure in the context of short interest-rate models).

We then find that

$$di = \left[\frac{d\varphi}{dt} - \alpha_{\rm I}[i-\varphi(t),t](i-\varphi(t))\right]dt + \sigma_{\rm I}[i-\varphi(t),t]dW_{\rm I}. \tag{58}$$

The price of a zero-coupon Inflation-linked bond is given by,

$$\begin{aligned} P_I(t_0,T) &= \mathbb{E}\left[e^{\int_{t_0}^{T}[i(t')-r(t')]dt'}|r(t_0),i(t_0)\right] & (59)\\ &= e^{\int_{t_0}^{T}\varphi(t')dt'}P_I^x(t_0,T;x(t_0)). & (60)\end{aligned}$$

We can then extract the correct function $\varphi(t)$:

$$\varphi(t) = \frac{\partial \ln P_I(0,t)}{\partial t} - \frac{\partial \ln P_I^x(0,t;x(0))}{\partial t}. \tag{61}$$

In general there are no analytic solutions for $P_I^x(0,t)$, but we can find it to the required accuracy by Monte Carlo sampling. Therefore the Monte Carlo pricing of an inflation product in this general model required a preliminary run to find $P_I^x(0,t)$, and then a second run to price the product using the correct $\varphi(t)$.

At this stage the model is too general, so we make a restricted choice of the parameters, which will allow for the kind of market smile seen:

$$\begin{aligned} \alpha_{\rm I} &= \begin{cases} \alpha_- & \text{for } x < -\xi \\ \alpha_0 & \text{for } -\xi \le x \le \xi \\ \alpha_+ & \text{for } \xi < x, \end{cases} \\ \sigma_{\rm I} &= \begin{cases} \sigma_- & \text{for } x < -\xi \\ \sigma_0 & \text{for } -\xi \le x \le \xi \\ \sigma_+ & \text{for } \xi < x. \end{cases} & (62)\end{aligned}$$

The idea is that the dynamics changes when the inflation rate drifts beyond a certain range, ξ, from its mean-reversion level. This could be understood in terms of modern monetary policy where the central banks try to keep inflation within a fixed range. As long as the rate is within this range, it displays regular behaviour understood by the markets. However, there is much greater market uncertainty on how rates behave when they go outside of this range, leading to perhaps larger volatility. (Such interpretations need a pinch of salt: the

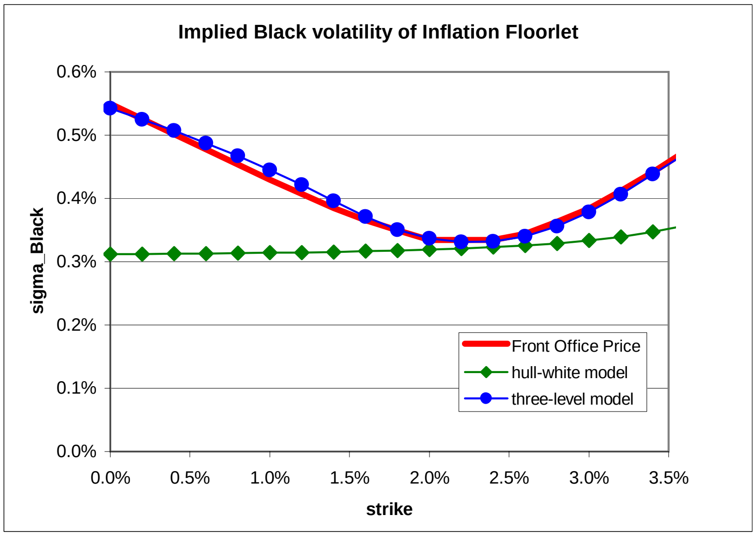

Figure 5: Results of Monte Carlo pricing of a ten-year inflation floorlet using the three-level volatility structure defined in (62). Plotted is the implied Black volatility from this price (blue circles), using parameter values to fit to the RBS front-office price (red line). The values are $\alpha_- = \alpha_0 = \alpha_+ = 0.1$, $\sigma_- = 1.5\%$, $\sigma_0 = 0.32\%$, $\sigma_+ = 1.6\%$, and $\xi = 1\%$. The green diamonds are the result of the Hull-White inflation model with constant parameters, $\alpha_{\rm I} = 0.1$ and $\sigma_{\rm I} = 0.48\%$ (The implied volatility is not completely flat here because of the convexity correction).

strong smile in implied vols from inflation caps/floors have a lot to do with supply and demand, which cannot be included in our model.)

With this model for the volatility structure, we just need to play with seven parameters to try and fit to market prices (actually, if the times of the product are fixed, we generally only need to adjust σ_x and not α_x, so there are only four parameters to play with). In figure 5 we show the price of inflation floors as a function of the strike, compared to the price that the RBS front office supplies. We see that it is possible to capture a strong smile with this model, in contrast to the simple Hull-White inflation model with flat parameters.

5 Interest Rates and Inflation Hybrids

The simplest rates-inflation hybrids are swaps of inflation related payouts with floating payouts. As the total value can be written as the sum of the two legs, there is not much new in the pricing of these products.

On the other hand there are more exotic products traded which must be handled with the combined stochastic processes for rates and for inflation. As a concrete example we consider an interest-rate caplet where the cap is related to the inflation rate. This has a payoff at time T_2 of,

$$\text{payoff} = (T_2 - T_1)\left(f(T_1,T_2) + K - R\left(\frac{I(T_2-\tau_{\rm delay})}{I(T_2-\tau_{\rm delay}-1)} - 1\right)\right)_+ \tag{63}$$

where K is a spread and R an inflation gearing. The forward rate $f(T_1,T_2)$ is the floating rate (LIBOR rate) for lending at time T_1 for a time of $T_2 - T_1$. Note that the inflation rate is defined as the average rate over the year ending at $T_2 - \tau_{\rm delay}$. By averaging over a year, we remove seasonal effects, and the time delay allows for publication of the index.

We will now show how to price this inflation-dependent cap on the forward rate. First we note that $f(T_1, T_2) = (1 - P(T_1, T_2))/\tau P(T_1, T_2)$, which is a path dependent quantity (unknown at earlier time t). Fortunately, in the Hull-White model there is a closed form result for $P(T_1, T_2)$ when we know the short rate $r(T_1)$. It is given by[6],

$$P(T_1, T_2; r(T_1)) = \exp[A(T_1, T_2) - r(T_1)B(\alpha, T_2 - T_1)] \tag{64}$$

with $A(T_1, T_2)$ a deterministic function that depends also on the current market discount curve $P(t, T)$. Therefore, writing the value of this product as the discounted expectation of the payoff we get in the two-process Hull-White model,

$$v_{\text{irHybrid}} = \mathbb{E}\left[\left(e^{-A(T_1,T_2)+r(T_1)B(\alpha,T_2-T_1)} - 1 + \tau(K + R) - \tau R e^{\int_{T_2-1}^{T_2} i(t')dt'}\right)_+ e^{-\int_t^{T_2} r(t')dt'}\right], \tag{65}$$

where we have set τ_{delay} to zero for simplicity.

We rewrite the price as,

$$v_{\text{irHybrid}} = \mathbb{E}[(ae^x + be^y + c)_+ e^z] \tag{66}$$

where x, y and z are all correlated Gaussian distributed random variables. Explicitly, we have,

$$\begin{aligned}
a &= -R\tau \\
b &= e^{-A(T_1,T_2)} \\
c &= (R + K_2)\tau - 1 \\
x &= \int_{T_2-1}^{T_2} i(t')dt' \\
y &= r(T_1)B(\alpha_r, T_2 - T_1) \\
z &= -\int_t^{T_2} r(t')dt'
\end{aligned} \tag{67}$$

The means and variances of x, y and z are given in appendix A, and can be used to evaluate the expectation by Monte Carlo sampling. Alternatively we can use a short-stepping version of Monte Carlo to generate paths of $i(t)$ and $r(t)$ and then evaluate the expectation, without needing the variances in appendix A.

A faster approximate method with a closed-form result can also be used if the variances are small: We replace $ae^x + be^y$ with a normal distribution, such that the mean and variance are matched. We then find the price of the hybrid to be:

$$v_{\text{irHybrid}} \approx P(t, T_2)\left\{f(q/\sqrt{V}) + q\Phi[q/\sqrt{V}]\right\} \tag{68}$$

where $f(x)$ is the normal function,

$$V = a^2 e^{2x_0+v_x}(e^{v_x} - 1) + b^2 e^{2y_0+v_y}(e^{v_y} - 1) + 2abe^{x_0+y_0+(v_x+v_y)/2}(e^{v_{xy}} - 1) \tag{69}$$

and

$$q = \frac{1}{P(t, T_2)}\left[P(t, T_1) - RP(t, T_2 - 1)\tau \frac{P_I(t, T_2)}{P_I(t, T_2 - 1)} e^{C(t,T_2-1,T_2)}\right] + \tau(R + K_2) - 1. \tag{70}$$

A comparison of the normal approximation with results from Monte Carlo sampling is shown in figure 6, where we see that for realistic parameters (i.e., small enough volatility) the normal approximation is extremely accurate.

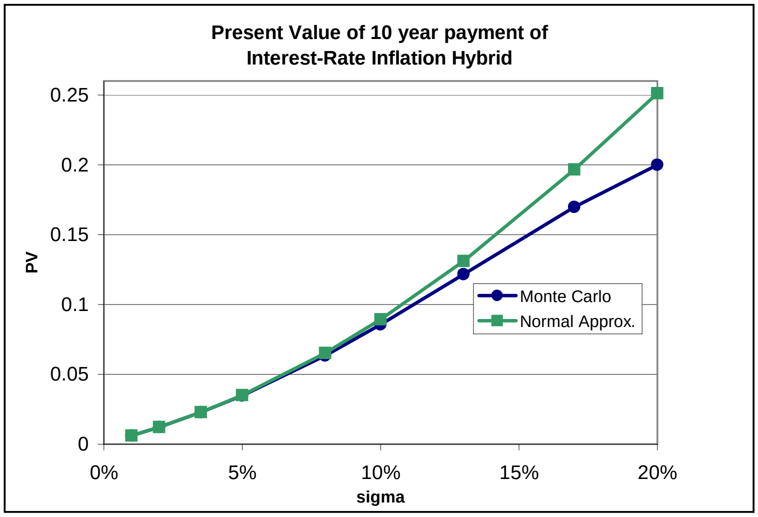

Figure 6: The present value of the payment of (63) in ten years, calculated within the Hull-White inflation model for different values of σ and $\sigma_{\rm I}$. The blue circles are results from Monte Carlo sampling and the green squares are the analtyical result of the normal approximation (68). For realistic parameters we usually have $\sigma, \sigma_{\rm I} < 2\%$, and so the normal approximation is extremely accurate.

6 Conclusion

To summarize, we have introduced inflation-linked derivatives, and shown that they can be priced using a relatively simple two-process short-rate model. On the other hand, simple models with constant parameters do not capture the volatility-smile seen in market prices for inflation options. We can do this by generalizing to an inflation short-rate model with local volatility, which can be used for pricing by Monte Carlo sampling. The two-process short-rate models naturally include the capability to price interest-rate inflation hybrids, and we have shown how to do this for a caplet with inflation-linked strike.

We end this lecture with an indication of how one could price other hybrids by adding extra processes to the model. An increasingly popular retail product is a bond that pays the maximum of an equity index or a price index. In fact it was the emergence of this product in Italy that was a big driver of the Eurozone inflation swaps market (to allow banks to hedge their inflation exposure on these products). The payoff at maturity T of this bond could take the form,

$$\text{payoff} = N \left[\frac{S(T)}{S(T_0)} - 1 + \left(\frac{I(T)}{I(T_0)} - \frac{S(T)}{S(T_0)} \right)_+ \right]. \tag{71}$$

In fact, products on the market tend to be more complicated: e.g., the above expression could be negative, so the payment is usually floored at zero, and also capped at some maximum value. In addition, the equity payoff is typically proportional to an average of the index over several previous dates.

To model the price of such products we will need to include another process for the equity index. This must be correlated to the inflation and interest rate processes. For example, we could model an equity index that has the value $S(t)$ at time t by a log-normal process with local volatility:

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma_{\rm E}(S)dW_{\rm E}(t), \tag{72}$$

where in the risk-neutral measure, $r(t)$ is the short interest rate which obeys the process in (22). The form of the local volatility $\sigma_{\mathrm{E}}(S)$ should be determined by the prices of vanilla options on the equity index in the usual way. Perhaps the most problematic part of pricing this kind of hybrid is the choice of correlation parameter, at least until there exists a liquid market in such hybrids.

A Means and variances for a Rates-Inflation hybrid

We will need the means, variances and covariances of the three random variables, x, y and z defined in (67).

A.1 Variances

$$\begin{aligned} v_x &= \mathrm{Var}\left[\int_{T_i-1}^{T_i} i(t')dt'|r(t)\right] \\ &= \frac{\sigma_i^2}{\alpha_i^2}\left[1 - B(\alpha_i,1) - \frac{\alpha_i}{2}B(\alpha_i,1)^2\right] \\ &\quad +\sigma_i^2 B(2\alpha_i, T_i-1-t)B(\alpha_i,1)^2 \end{aligned} \tag{73}$$

$$\begin{aligned} v_y &= B^2(\alpha_r, T_i - T_{i-1})\mathrm{Var}\left[r(T_{i-1})|r(t)\right] \\ &= \sigma_r^2 B^2(\alpha_r, T_i - T_{i-1})B(2\alpha_r, T_{i-1}-t). \end{aligned} \tag{74}$$

$$\begin{aligned} v_z &= \mathrm{Var}\left[\int_{t}^{T_i} r(t')dt'|r(t)\right] \\ &= \frac{\sigma_r^2}{\alpha_r^2}\left[(T_i-t) - B(\alpha_r, T_i-t) - \frac{\alpha_r}{2}B(\alpha_r, T_i-t)^2\right] \end{aligned} \tag{75}$$

A.2 Covariances

$$\begin{aligned} v_{xz} &= -\mathrm{Covar}\left[\int_{T_i-1}^{T_i} i(t')dt', \int_{t}^{T_i} r(t')dt'|i(t),r(t)\right] \\ &= -\rho\sigma_i\sigma_r\left\{\frac{1}{\alpha_i\alpha_r}\right. \\ &\quad -\frac{1}{(\alpha_i+\alpha_r)}\left(\frac{1}{\alpha_i}B(\alpha_i,1) + B(\alpha_i,1)B(\alpha_r,1) + \frac{1}{\alpha_r}B(\alpha_r,1)\right) \\ &\quad +B(\alpha_i,1)B(\alpha_r,1)B(\alpha_i+\alpha_r, T_i-1-t) \\ &\quad \left.+\frac{1}{\alpha_r}B(\alpha_i,1)[B(\alpha_i,T_i-1-t) - B(\alpha_i+\alpha_r, T_i-1-t)]\right\}. \end{aligned} \tag{76}$$

$$v_{xy} = B(\alpha_r, T_i - T_{i-1})\mathrm{Covar}\left[\int_{T_i-1}^{T_i} i(t')dt', r(T_{i-1})|i(t),r(t)\right]$$

$$
\begin{aligned}
&= \rho\sigma_i\sigma_r B(\alpha_r, T_i - T_{i-1}) \Big\{ B(\alpha_i + \alpha_r, T_i - 1 - t) B(\alpha_i, 1) e^{-\alpha_r(T_{i-1} - T_i + 1)} \\
&\quad + \frac{1}{\alpha_i} \Big[B(\alpha_r, T_{i-1} - T_i + 1) - e^{-\alpha_i(T_i - T_{i-1})} B(\alpha_i + \alpha_r, T_{i-1} - T_i + 1) \Big] \Big\} \quad (77)
\end{aligned}
$$

$$
\begin{aligned}
v_{yz} &= -B(\alpha_r, T_i - T_{i-1}) \text{Covar} \left[\int_t^{T_i} r(t')dt', r(T_{i-1}) | i(t), r(t) \right] \\
&= \frac{\sigma_r^2}{\alpha_r} B(\alpha_r, T_i - T_{i-1}) \Big[B(\alpha_r, T_{i-1} - t) - B(2\alpha_r, T_{i-1} - t) e^{-\alpha_r(T_i - T_{i-1})} \Big]. \quad (78)
\end{aligned}
$$

A.3 Means

To get the mean of z we use the fact that $\mathbb{E}[e^z] = e^{z_0 + v_z/2} = P(t, T_i)$ so that

$$
\begin{aligned}
z_0 &= \text{Mean} \left[-\int_t^{T_i} r(t')dt' | r(t) \right] \\
&= -\frac{v_z}{2} + \ln P(t, T_i) \\
&= \ln P(t, T_i) - \frac{\sigma_r^2}{2\alpha_r^2} \Big[(T_i - t) - B(\alpha_r, T_i - t) - \frac{\alpha_r}{2} B(\alpha_r, T_i - t)^2 \Big] \quad (79)
\end{aligned}
$$

For y we have

$$
\begin{aligned}
y_0 &= B(\alpha_r, T_i - T_{i-1}) \text{Mean} \left[r(T_{i-1}) | r(t) \right] \\
&= B(\alpha_r, T_i - T_{i-1}) \left[f(t, T_{i-1}) + \frac{1}{2} \sigma^2 B(\alpha_r, T_{i-1} - t)^2 \right] \quad (80)
\end{aligned}
$$

To get the mean of x we use the fact that $\mathbb{E}[e^{x+z}] = e^{x_0 + z_0 + \frac{1}{2}v_x + \frac{1}{2}v_z + v_{xz}} = v_{\text{Iswap}}$ where v_{Iswap} is the value of a year-on-year inflation swap, given by (see section 3.3.3):

$$
v_{\text{Iswap}} = P(t, T_i - 1) \frac{P_I(t, T_i)}{P_I(t, T_i - 1)} e^{C(t, T_i - 1, T_i)} \quad (81)
$$

where C is the covariance,

$$
\begin{aligned}
C(t, T_i - 1, T_i) &= \text{Covar} \left[\int_t^{T_i - 1} i, \int_{T_i - 1}^{T_i} (r - i) \right] \\
&= -\frac{\sigma_i^2}{2} B^2(\alpha_i, T_i - 1 - t) B(\alpha_i, 1) \\
&\quad + \frac{\rho\sigma_i\sigma_r}{\alpha_i} B(\alpha_r, 1) [B(\alpha_r, T_i - t - 1) - B(\alpha_i + \alpha_r, T_i - t - 1)]. \quad (82)
\end{aligned}
$$

so that

$$
\begin{aligned}
x_0 &= \text{Mean} \left[\int_{T_i - 1}^{T_i} i(t')dt' | r(t), i(t) \right] \\
&= -\left(z_0 + \frac{1}{2} v_x + \frac{1}{2} v_z + v_{xz} \right) + \ln v_{\text{Iswap}} \\
&= -\left(\frac{1}{2} v_x + v_{xz} \right) + \ln v_{\text{Iswap}} - \ln P(t, T_i) \\
&= -\left(\frac{1}{2} v_x + v_{xz} \right) + \ln \left[\frac{P(t, T_i - 1)}{P(t, T_i)} \frac{P_I(t, T_i)}{P_I(t, T_i - 1)} \right] + C(t, T_i - 1, T_i) \quad (83)
\end{aligned}
$$

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